Multiple scattering temporal correlation function in a half space with finite-size heterogeneities

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An exact solution for the boundary problem of temporal correlations of light multiply scattered from a medium occupying a half space is found by means of the Wiener-Hopf method, taking into account single-scattering anisotropy. Within the P_1 approximation a universal initial decay rate of the temporal correlation function is obtained. For larger time intervals a higher single-scattering anisotropy yields a higher decay rate contrary to predictions of the diffusion approximation. Within the P_2 approximation, which takes account of the first- and second-order Legendre polynomials, the solution obtained becomes universal in an expanded temporal range and agrees rather well with the known measurement data.

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I. INTRODUCTION

Beginning with the discovery of photon weak localization [1,2] much attention has been paid to studies of multiple light scattering from highly inhomogeneous media (see, e.g., [3–6]). For most problems of multiple scattering, boundedness of the scattering system appears to be essential. Within the diffusion approximation the mirror image method is widely used [7], satisfying mixed Dirichlet boundary conditions. However, the mirror image method appears to be insufficient [8] for a description of coherent backscattering or of the intensity correlation functions of reflected or transmitted light, since these phenomena are caused by light transport through a surface layer wherein the approximation valid for an infinite medium becomes inadequate.

To account for the finite size of the scatterers one resorts to an expansion in spherical harmonics [9,10]. The diffusion approximation implies that such an expansion is restricted to the first-order, or P_1, Legendre polynomial. Within this P_1 approximation the transport mean free path l^* becomes the characteristic scale [11] instead of the photon mean free path l due to the single-scattering anisotropy. Contributions of higher order spherical harmonics were considered in Refs. [12,13] going beyond the diffusion approximation for an infinite medium also. For a system of pointlike scatterers there exists the famous exact Milne solution of the boundary problem of multiple scattering from a half space [9,10,14]. This solution was generalized in Refs. [15,16] to include the coherent backscattering. In Refs. [6,17] the limit of highly anisotropic single scattering, $\overline{\cos\theta}\to 1$, where $\overline{\cos\theta}$ is the mean cosine of the single scattering angle, was shown to be exactly soluble.

Here, generalizing the Milne approach, we solve the boundary problem for the temporal correlation function of multiple scattering from a system of Brownian particles, taking the single-scattering anisotropy into account. We consider the case of moderate anisotropy, permitting the Legendre polynomial analysis, and find an exact solution, which explicitly exhibits a dependence on the parameters of anisotropy $\overline{\cos\theta}$ and $\overline{\cos^2\theta}$. The solution obtained permits calculation of the intensity correlation function in a wide temporal range. Usually for the initial temporal range one assumes a decay law of the form $1-\gamma\sqrt{6t/\tau}$ for the field correlation function, where t is the time, $\tau = 1/Dk^2$ is the characteristic time, D is the self-diffusion coefficient of a Brownian particle, k is the wave number, and the parameter γ describes the initial slope. It was found in Refs. [18,19] that the initial slope γ approximately equal to 2 is universal, independent of the concentration and size of the scattering particles. In contrast to these results the diffusion approximation is known to predict a specific dependence on the single-scattering anisotropy. In particular, the initial slope varies from the value $\gamma = 2.4$ for the isotropic phase function, $\overline{\cos\theta}=0$, to $\gamma = 0.71$ for $\overline{\cos\theta}\to 1$ [8]. In the present paper by solving the boundary problem in the P_1 approximation we find that the initial slope γ is exactly 2. For larger time values the decay rate of the correlation function grows with increasing $\overline{\cos\theta}$, quite opposite to the prediction of the diffusion approximation. Within the P_2 approximation we find that the dependence on the parameters of anisotropy becomes much weaker and the intensity correlation function calculated within the P_2 approximation agrees rather well with the known measurements [19] in a wide range of its variation.

The paper is organized as follows. In Sec. II we derive the Bethe-Salpeter equation for the coherence function. In Sec. III, by expanding the coherence function into a Legendre polynomial series, an equation set is obtained for the terms of the expansion, and its solution is found using the Wiener-Hopf method. In Sec. IV the results of calculations of the temporal correlation function within the P_1 and P_2 approximations are given and analyzed. In the Appendix the derivation of a Milne-like solution in the framework of the Wiener-Hopf method is described in detail.

II. BETHE-SALPETER EQUATION FOR A MEDIUM OCCUPYING A HALF SPACE

We consider a medium with random permittivity $\varepsilon(\mathbf{r},t) = \varepsilon + \delta\varepsilon(\mathbf{r},t)$ which fluctuates in space and time about the ensemble average $\varepsilon = \langle\varepsilon(\mathbf{r},t)\rangle$. Neglecting polarization effects we change the Maxwell wave equation into the Helmholz one and present it in the form

$$E(\mathbf{r},t)=\langle E(\mathbf{r})\rangle+\frac{1}{4\pi}\int d\mathbf{r}_1 T(\mathbf{r}-\mathbf{r}_1)\,\delta\varepsilon(\mathbf{r}_1,t)E(\mathbf{r}_1,t), \tag{1}$$

where $\langle E(\mathbf{r})\rangle=E\exp(i\mathbf{k}_i\cdot\mathbf{r}-i\omega t)$ is the mean field, $\mathbf{k}_i$ is the incident wave vector in the medium, $T(\mathbf{r})=k_0^2\exp(ikr)/r$ is the Green's function of the scalar wave equation, and $k_0=\omega/c=2\pi/\lambda$ is the vacuum wave number. We omit the monochromatic factor $\exp(i\omega t)$ since the time it takes the light to propagate through the medium is sufficiently less than the characteristic interval of the random permittivity variations.

We define the scattered field as $\delta E(\mathbf{r},t)=E(\mathbf{r},t)-\langle E(\mathbf{r},t)\rangle$. An average of the product of two complex-conjugated fields $\delta E^*(\mathbf{r},t)$ and $\delta E(\mathbf{r},0)$ at different moments of time can be presented in the form

$$\langle\,\delta E^*(\mathbf{r},t)\,\delta E(\mathbf{r},0)\rangle=r^{-2}E^2C(t|\mathbf{k}_s,\mathbf{k}_i),$$

where $C(t|\mathbf{k}_s,\mathbf{k}_i)$ is the temporal field correlation function

$$C(t|\mathbf{k}_s,\mathbf{k}_i)=\int d\mathbf{r}_2d\mathbf{r}_2'd\mathbf{r}_1d\mathbf{r}_1'\exp(-i\mathbf{k}_s\cdot\mathbf{r}_2+i\mathbf{k}_s^*\cdot\mathbf{r}_2' +i\mathbf{k}_i\cdot\mathbf{r}_1-i\mathbf{k}_i^*\cdot\mathbf{r}_1')\Gamma(\mathbf{r}_2,\mathbf{r}_2',\mathbf{r}_1,\mathbf{r}_1',t). \tag{2}$$

Here r is the distance from the scattering system to the observation point, much exceeding any other spatial length, and $\mathbf{k}_s$ is the scattered wave vector. The function $\Gamma(\mathbf{r}_2,\mathbf{r}_2',\mathbf{r}_1,\mathbf{r}_1',t)$, known as the ladder propagator, satisfies the Bethe-Salpeter equation

$$\Gamma(\mathbf{r}_2,\mathbf{r}_2',\mathbf{r}_1,\mathbf{r}_1',t)=k_0^4G(\mathbf{r}_2-\mathbf{r}_2',t)\Big[\delta(\mathbf{r}_2-\mathbf{r}_1)\,\delta(\mathbf{r}_2'-\mathbf{r}_1') +\int d\mathbf{r}_3d\mathbf{r}_3'T(\mathbf{r}_2-\mathbf{r}_3) \times T^*(\mathbf{r}_2'-\mathbf{r}_3')\Gamma(\mathbf{r}_3,\mathbf{r}_3',\mathbf{r}_1,\mathbf{r}_1',t)\Big], \tag{3}$$

where

$$G(\mathbf{r}_1-\mathbf{r}_2,t_1-t_2)=\frac{1}{(4\pi)^2}\langle\,\delta\varepsilon(\mathbf{r}_1,t_1)\,\delta\varepsilon(\mathbf{r}_2,t_2)\rangle \tag{4}$$

is the binary correlator of permittivity fluctuations.

Assuming that the temporal decay of the permittivity fluctuation can be described as Brownian particle diffusion, one gets for the Fourier transform of the permittivity correlator

$$\tilde{G}(\mathbf{q},t)=\int d\mathbf{r}G(\mathbf{r},t)\exp(-i\mathbf{q}\cdot\mathbf{r})=\tilde{G}_0(\mathbf{q})\exp(-t/\tau). \tag{5}$$

The function $\tilde{G}_0(\mathbf{q})=\tilde{G}(\mathbf{q},0)$ is the single-scattering cross section, or phase function. We consider the weak scattering regime, $\lambda\ll l$.

Let the medium occupy the half space $z>0$, where z is the Cartesian coordinate normal to the boundary. Introducing the center-of-mass coordinates $\mathbf{R}_j=(\mathbf{r}_j+\mathbf{r}_j')/2$ and the relative ones $\mathbf{r}_j''=\mathbf{r}_j-\mathbf{r}_j'$, one presents the exponential entering Eq. (2) as follows:

$$\exp(-i\mathbf{k}_s\cdot\mathbf{r}_2+i\mathbf{k}_s^*\cdot\mathbf{r}_2'+i\mathbf{k}_i\cdot\mathbf{r}_1-i\mathbf{k}_i^*\cdot\mathbf{r}_1') \approx\exp\left(-\frac{z_1}{l\cos\theta_i}+\frac{z_2}{l\cos\theta_s}\right)\exp(i\mathbf{k}_i\cdot\mathbf{r}_1''-i\mathbf{k}_s\cdot\mathbf{r}_2''), \tag{6}$$

where θ_i and θ_s are the angles of incidence and scattering, respectively. Equation (3) can be presented in the form

$$\Gamma(\mathbf{R}_2,\mathbf{R}_1,t|\mathbf{k}_s,\mathbf{k}_i)=k_0^4\tilde{G}(\mathbf{k}_s-\mathbf{k}_i,t)\,\delta(\mathbf{R}_2-\mathbf{R}_1) +k_0^4\int d\mathbf{R}_3\tilde{G}(-\mathbf{k}_s+\mathbf{k}_{23},t) \times\Lambda(R_{23})\Gamma(\mathbf{R}_3,\mathbf{R}_1,t|\mathbf{k}_{23},\mathbf{k}_i), \tag{7}$$

where

$$\Gamma(\mathbf{R}_2,\mathbf{R}_1,t|\mathbf{k}_s,\mathbf{k}_i)=\int d\mathbf{r}_1''d\mathbf{r}_2''\Gamma\left(\mathbf{R}_2+\frac{\mathbf{r}_2''}{2},\mathbf{R}_2-\frac{\mathbf{r}_2''}{2}, \mathbf{R}_1+\frac{\mathbf{r}_1''}{2},\mathbf{R}_1-\frac{\mathbf{r}_1''}{2},t\right) \times\exp(-i\mathbf{k}_s\cdot\mathbf{r}_2''+i\mathbf{k}_i\cdot\mathbf{r}_1'') \tag{8}$$

is the Fourier transform of the ladder operator with respect to the relative coordinates, $\mathbf{k}_{ij}=k\mathbf{R}_{ij}R_{ij}^{-1}$ is the wave vector propagating from $\mathbf{R}_j$ to $\mathbf{R}_i$, $\mathbf{R}_{ij}=\mathbf{R}_i-\mathbf{R}_j$, and the propagator $\Lambda(R)=R^{-2}\exp(-R/l)$ stems from the product of two complex-conjugated Green's functions $T(\mathbf{R})$ and $T^*(\mathbf{R})$, $k_0^4\Lambda(R)=T(\mathbf{R})T^*(\mathbf{R})$.

The ladder operator depends, due to the geometry considered, on the relative two-dimensional vector $\rho_{21}=(\mathbf{R}_2-\mathbf{R}_1)_\perp$, lying in the (x,y) plane tangential to the boundary, and the coordinates z_1 and z_2:

$$\Gamma(\mathbf{R}_2,\mathbf{R}_1,t|\mathbf{k}_s,\mathbf{k}_i)=\Gamma(\boldsymbol{\rho}_{21},z_2,z_1,t|\mathbf{k}_s,\mathbf{k}_i). \tag{9}$$

Applying the integral operation $\int_0^\infty dz_1\exp[-z_1/(l\cos\theta_i)]$ to Eq. (8) we get

$$\Phi(\boldsymbol{\rho}_{21},z_2,t|\mathbf{k}_s,\mathbf{k}_i)=k_0^4\tilde{G}(\mathbf{k}_s-\mathbf{k}_i,t)\,\delta(\boldsymbol{\rho}_{21})\,\theta(z_2)\exp(-z_2/l) +k_0^4\int\tilde{G}(-\mathbf{k}_s+\mathbf{k}_{23},t)\Lambda(R_{23}) \times\Phi(\boldsymbol{\rho}_{31},z_3,t|\mathbf{k}_{23},\mathbf{k}_i)d\mathbf{R}_3, \tag{10}$$

where

$$\Phi(\boldsymbol{\rho}_{21},z_2,t|\mathbf{k}_s,\mathbf{k}_i)=\int_0^\infty dz_1\exp[-z_1/(l\cos\theta_i)] \times\Gamma(\boldsymbol{\rho}_{21},z_2,z_1,t|\mathbf{k}_s,\mathbf{k}_i). \tag{11}$$

Equation (10) can be considered as the Schwartzschild-Milne equation generalized for the temporal correlations.

The temporal field correlation function (2) can be presented as

$$C(t|\mathbf{k}_s,\mathbf{k}_i)=S\int d\boldsymbol{\rho}_{21}\int_0^\infty dz_2\,\Phi(\boldsymbol{\rho}_{21},z_2,t|\mathbf{k}_s,\mathbf{k}_i)\times\exp[z_2/(l\cos\theta_s)], \tag{12}$$

where S is the illuminated area. Recall that $\theta_s>\pi/2$ in the backscattering geometry and the exponential in Eq. (12) vanishes at large z_2.

III. THE GENERALIZED MILNE EQUATION SOLUTION

The function $\Phi(\boldsymbol{\rho}_{21},z_2,t|\mathbf{k}_s,\mathbf{k}_i)$ depends on the directions of the three vectors $\boldsymbol{\rho}_{21}$, $\mathbf{k}_s$, and $\mathbf{k}_i$. For simplicity we restrict ourselves to the case of normal incidence with vector $\mathbf{k}_i$ directed along the unit vector $\mathbf{e}_z$ normal to the boundary, $\mathbf{k}_i=k(0,0,1)$. In a spherical coordinate system the vectors $\mathbf{k}_s$ and $\boldsymbol{\rho}_{21}$ can be parametrized as

$$\mathbf{k}_s=k(\sin\theta_s\cos\phi_s,\sin\theta_s\sin\phi_s,\cos\theta_s),$$

$$\boldsymbol{\rho}_{21}=\rho_{12}(\cos\phi_\rho,\sin\phi_\rho,0).$$

We expand the function $\Phi(\boldsymbol{\rho}_{21},z_2,t|\mathbf{k}_s,\mathbf{k}_i)$ into a series in the spherical functions up to terms of the second order:

$$\begin{aligned}\Phi(\boldsymbol{\rho}_{21},z_2,t|\mathbf{k}_s,\mathbf{k}_i)=\frac{1}{4\pi l}[&\gamma_0(\boldsymbol{\rho}_{21},z_2,t)P_0^0(\cos\theta_s)\\&+\gamma_1(\boldsymbol{\rho}_{21},z_2,t)P_1^0(\cos\theta_s)\\&+\gamma_1^{(1)}(\boldsymbol{\rho}_{21},z_2,t)P_1^1(\cos\theta_s)\cos\phi\\&+\gamma_2(\boldsymbol{\rho}_{21},z_2,t)P_2^0(\cos\theta_s)\\&+\gamma_2^{(1)}(\boldsymbol{\rho}_{21},z_2,t)P_2^1(\cos\theta_s)\cos\phi\\&+\gamma_2^{(2)}(\boldsymbol{\rho}_{21},z_2,t)P_2^2(\cos\theta_s)\cos2\phi],\end{aligned} \tag{13}$$

where $P_i^j(x)$ are the Legendre polynomials: $P_0^0(x)=1$, $P_1^0(x)=x$, $P_1^1(x)=\sqrt{1-x^2}$, $P_2^0(x)=(3x^2-1)/2$, $P_2^1(x)=3x\sqrt{1-x^2}$, $P_2^2(x)=3(1-x^2)$, and $\phi=\phi_s-\phi_\rho$.

To calculate the temporal correlation function one needs to integrate $\Phi(\boldsymbol{\rho}_{21},z_2,t|\mathbf{k}_s,\mathbf{k}_i)$ over tangential coordinates $\boldsymbol{\rho}_{21}$, as is seen from Eq. (12). Therefore terms containing the angle ϕ in the series (13) do not contribute to the temporal correlation function for symmetry reasons, and can be omitted.

Performing successively in Eq. (10) convolutions of the forms $\int d\boldsymbol{\rho}_{21}\int d\Omega_s$, $\int d\boldsymbol{\rho}_{21}\int d\Omega_s P_1^0(\cos\theta_s)$, and $\int d\boldsymbol{\rho}_{21}\int d\Omega_s P_2^0(\cos\theta_s)$, where $\int d\Omega_s$ denotes integration over orientations of the vector $\mathbf{k}_s$, we obtain the equation set

$$\frac{1}{2j+1}\gamma_j(z_2,t)=g_j\Bigg[\theta(z_2)\exp(-z_2/l)+\frac{1}{4l}\sum_{k=0,1,2}\int_0^\infty dz_3\Lambda_{jk}(0,z_{23})\gamma_k(z_3,t)\Bigg] \tag{14}$$

for $j=1,2,3$, where

$$\gamma_j(z,t)=\int\gamma_j(\boldsymbol{\rho}_{21},z,t)d\boldsymbol{\rho}_{21}. \tag{15}$$

The parameters

$$g_j=g_j(t)=lk_0^4\int d\Omega_s\tilde{G}(\mathbf{k}_s-\mathbf{k}_i,t)P_j^0(\cos\theta_s)=\overline{P_j^0(\cos\theta)\exp[-2(1-\cos\theta)t/\tau]} \tag{16}$$

describe the evolution of the binary permittivity correlator expanded in Legendre polynomials. It is worth noting that the parameters g_1 and g_2 do not become zero at $t\neq0$ even in the case of an isotropic phase function. In this case these parameters can easily be found:

$$g_0=\frac{\exp(-2t/\tau)\sinh(2t/\tau)}{2t/\tau},$$

$$g_1=\frac{\exp(-2t/\tau)[(2t/\tau)\cosh(2t/\tau)-\sinh(2t/\tau)]}{(2t/\tau)^2}. \tag{17}$$

When $t/\tau\ll1$ the problem of the temporal dependence of the correlation function is formally identical to that of the scattering intensity for albedo less than unity. However, with increasing t, these problems are seen to become different since terms g_j, $j\geqslant1$, can no longer be ignored. For an arbitrary phase function the parameters $g_j(t)$ can be written at small time $t<\tau$ as follows:

$$g_0\approx1-\frac{2t}{\tau}(1-\overline{\cos\theta}),$$

$$g_1\approx\overline{\cos\theta}-\frac{2t}{\tau}(\overline{\cos\theta}-\overline{\cos^2\theta}),$$

etc. The optical theorem relates the permittivity correlator to the photon mean free path:

$$l^{-1}=k_0^4\int d\Omega_s\tilde{G}(\mathbf{k}_s-\mathbf{k}_i,0). \tag{18}$$

The symmetric elements $\Lambda_{jk}(0,z)$ are defined as follows:

$$\Lambda_{00}(0,z)=2\int_1^\infty\frac{dr}{r}\exp(-|z|r/l),$$

$$\Lambda_{01}(0,z)=2\ \mathrm{sgn}(z)\int_1^\infty \frac{dr}{r^2}\exp(-|z|r/l),$$

$$\Lambda_{02}(0,z)=\int_1^\infty \frac{dr}{r}\left(\frac{3}{r^2}-1\right)\exp(-|z|r/l),$$

$$\Lambda_{11}(0,z)=2\int_1^\infty \frac{dr}{r^3}\exp(-|z|r/l),\tag{19}$$

$$\Lambda_{12}(0,z)=\ \mathrm{sgn}(z)\int_1^\infty \frac{dr}{r^2}\left(\frac{3}{r^2}-1\right)\exp(-|z|r/l),$$

$$\Lambda_{22}(0,z)=\frac{1}{2}\int_1^\infty \frac{dr}{r}\left(\frac{3}{r^2}-1\right)^2\exp(-|z|r/l).$$

We define the Laplace transforms of $\gamma_j(z,t)$ as follows:

$$\tilde{\gamma}_j(s,t)=\int_0^\infty \exp(-sz/l)\,\gamma_j(z,t)dz,\quad s\geqslant 0.\tag{20}$$

Performing the Laplace transformation of the integral equation set (14) with respect to the variable z_2, we get

$$[1-g_0m_0(s)]\tilde{\gamma}_0(s,t)+sm_1(s)g_0\tilde{\gamma}_1(s,t)-\frac{1}{2}g_0[3m_1(s)-m_0(s)]\tilde{\gamma}_2(s,t)=g_0a_0(s,t),$$

$$g_1sm_1(s)\tilde{\gamma}_0(s,t)+\left(\frac{1}{3}-g_1m_1(s)\right)\tilde{\gamma}_1(s,t)+\frac{1}{2}g_1s[3m_2(s)-m_1(s)]\tilde{\gamma}_2(s,t)=g_1a_1(s,t),\tag{21}$$

$$-\frac{1}{2}g_2[3m_1(s)-m_0(s)]\tilde{\gamma}_0(s,t)+\frac{1}{2}g_2s[3m_2(s)-m_1(s)]\tilde{\gamma}_1(s,t)+\left[\frac{1}{5}-\frac{1}{4}g_2[9m_2(s)-6m_1(s)+m_0(s)]\right]\tilde{\gamma}_2(s,t)=g_2a_2(s,t),$$

where

$$m_0(s)=\frac{1}{2s}\ln\left|\frac{1+s}{1-s}\right|,$$

$$m_1(s)=s^{-2}[m_0(s)-1],\quad m_2(s)=s^{-2}[m_1(s)-1/3],$$

$$a_0(s,t)=\frac{1}{1+s}-\frac{1}{2}\int_1^\infty \frac{ds_1}{s_1(s_1-s)}\tilde{\gamma}(s_1,t),$$

$$a_1(s,t)=\frac{1}{1+s}+\frac{1}{2}\int_1^\infty \frac{ds_1}{s_1^2(s_1-s)}\tilde{\gamma}(s_1,t),\tag{22}$$

$$a_2(s,t)=\frac{1}{1+s}-\frac{1}{4}\int_1^\infty \frac{ds_1}{s_1(s_1-s)}\left(\frac{3}{s_1^2}-1\right)\tilde{\gamma}(s_1,t).$$

We here define the function $\tilde{\gamma}(s,t)$ as follows:

$$\tilde{\gamma}(s,t)\equiv\tilde{\gamma}_0(s,t)-\frac{1}{s}\tilde{\gamma}_1(s,t)+\frac{1}{2}\left(\frac{3}{s^2}-1\right)\tilde{\gamma}_2(s,t).\tag{23}$$

Note that for $s\geqslant 1$ this function can be presented in the form

$$\tilde{\gamma}(s,t)=\tilde{\gamma}_0(s,t)-P_1(1/s)\tilde{\gamma}_1(s,t)+P_2(1/s)\tilde{\gamma}_2(s,t).\tag{24}$$

Thus the temporal field correlation function (12) in the P_2 approximation for backscattering, $\cos\theta_s<0$, takes the form

$$C(t|\mathbf{k}_s,\mathbf{k}_i)\sim\int d\boldsymbol{\rho}dz_2[\gamma_0(\boldsymbol{\rho},z_2,t)+\gamma_1(\boldsymbol{\rho},z_2,t)P_1^0(\cos\theta_s)+\gamma_2(\boldsymbol{\rho},z_2,t)P_2^0(\cos\theta_s)]\exp[z_2/(l\cos\theta_s)]=\tilde{\gamma}(s,t),\tag{25}$$

where $s=-1/\cos\theta_s$. As is seen from Eq. (25) the function $\tilde{\gamma}(1,t)$ determines the temporal correlation function of radiation scattered strictly backward, $\theta_s=\pi$,

$$\tilde{\gamma}(1,t)=C(t|-\mathbf{k}_i,\mathbf{k}_i)\equiv C(t),$$

less the coherent backscattering component, which is not considered here.

As is seen from Eq. (25), in deriving the temporal correlation function it is sufficient to find the function (23) rather than the Legendre expansion coefficients separately. Just this function $\tilde{\gamma}(s,t)$ enters the integral terms of the right-hand side (RHS's) of Eqs. (21). It turns out that Eqs. (21) can be rearranged in a linear combination that contains only the function $\tilde{\gamma}(s,t)$ and appears to be a closed Milne-like equation for this function, i.e., for the temporal correlation function. With this aim we multiply the second equation of the equation set (21) by $3s^{-1}(g_0-1)$ and the third one by $\frac{5}{2}[3s^{-2}(1-g_0)(1-g_1)-1]$ and add all three equations. We obtain

$$\psi(s,t)\tilde{\gamma}(s,t)=A(s,t),\tag{26}$$

where

$$\psi(s,t)=(1-g_0)\left[1-3g_1m_1(s)-\frac{15}{4}g_2(1-g_1)[3m_2(s)-m_1(s)]\right]-s^2\left[g_0m_1(s)-\frac{5}{4}g_2[3m_2(s)-m_1(s)]\right],\tag{27}$$

$$A(s,t)=g_0a_0-\frac{3}{s}(1-g_0)g_1a_1-\frac{5}{2}\left[1-\frac{3}{s^2}(1-g_0)\times(1-g_1)\right]g_2a_2. \tag{28}$$

Equation (26) is a closed integral equation with respect to the sought function $\tilde{\gamma}(s,t)$. At $t=0$ this function describes the multiple scattering intensity within the P_2 approximation. Putting $g_1=g_2=0$ one comes back to the Milne equation with the well-known solution.

Equation (26) exhibits analytic properties that permit us to apply to it the Wiener-Hopf method. This method is based on the regularity of functions entering the integral equation in the complex s plane. In fact, one requires these functions to be regular in some strip of the plane and simultaneously have a finite number of zeros there.

Describing details of the derivation in the Appendix, we present the solution as follows:

$$\tilde{\gamma}(s,t)=\frac{c_0(t)+c_1(t)s+c_2(t)s^2}{s^2(s+1)}\frac{\beta}{s^*+s}\exp[-J(s,t)], \tag{29}$$

where the parameters β and s^* are given exactly in Appendix. For $t\ll\tau$ they take the form

$$\beta\approx\sqrt{\frac{3}{1-g_2}},$$

$$s^*\approx\sqrt{3(1-g_0)(1-g_1)}\approx(1-\overline{\cos\theta})\sqrt{\frac{6t}{\tau}}. \tag{30}$$

The function $J(s,t)$ presents the integral

$$J(s,t)=\frac{s}{2\pi}\int_{-\infty}^{\infty}\frac{ds'}{s'^2+s^2}\ln\left[\frac{\beta^2\psi(is',t)}{s'^2+s^{*2}}\right] \tag{31}$$

and appears to be a generalization of the Chandrasekhar H function [15,20].

The parameters $c_0(t)$, $c_1(t)$, and $c_2(t)$, being constant with respect to s, can be found by substituting solution (29) into Eq. (26). Then expanding both sides of the equation into series in s and equating terms of the same order we get identities sufficient to determine $c_0(t)$, $c_1(t)$, and $c_2(t)$. Excess identities are then satisfied automatically. We also use the relationship [see Eq. (A15)]

$$c_0(t)-c_1(t)+c_2(t)=\frac{\beta}{1+s^*}\left\{g_0+3g_1(1-g_0)-\frac{5}{2}g_2[1-3(1-g_0)(1-g_1)]\right\}\times\exp[-J(1,t)], \tag{32}$$

which turns out to be necessarily satisfied because of the limits

$$(1+s)a_j(s,t)\to 1 \quad \text{with} \quad s\to -1, \quad j=0,1,2,$$

which follow from Eqs. (22).

The formula (29) presents the exact solution of the boundary problem for the temporal field correlation function in a medium occupying a half space, with the single-scattering anisotropy accounted for up to the P_2 terms.

In the P_1 approximation, at small $s\ll 1$ and $t\ll\tau$, one has

$$\psi(s,t)\approx 2(t/\tau)(1-\overline{\cos\theta})^2-s^2/3. \tag{33}$$

The reciprocal of this function describes the solution of the Bethe-Salpeter equation deep inside a medium, far from the boundary. Completed with the mirror image method, Eq. (33) describes the temporal correlation function in the diffusion approximation.

IV. CALCULATION RESULTS AND DISCUSSION

Using the solution obtained we calculate the temporal correlation function for isotropic and anisotropic single scattering in the P_1 and P_2 approximations. We restrict ourselves to the case of backward scattering, $\cos\theta_s=1$.

For isotropic single scattering the parameters $c_0(t)$ and $c_1(t)$ must vanish, $c_0(t)=c_1(t)=0$, and the remaining sole parameter $c_2(t)$ is determined by Eq. (32) as

$$c_2(t)=\beta g_0\exp[-J_M(1,t)](1+s^*)^{-1}, \tag{34}$$

where $J_M(s,t)=J(s,t)|_{g_1=g_2=0}$. For $s=1$ we get

$$\tilde{\gamma}_M(1,t)=\frac{3g_0(t)\exp[-2J_M(1,t)]}{2(1+s^*)^2}. \tag{35}$$

The subscript M indicates that the corresponding quantity relates to the isotropic Milne case, $g_1=g_2=0$. At $t=0$ one obtains the famous Milne solution

$$\tilde{\gamma}_M(s,0)=\frac{3\exp[-J_M(1,0)-J_M(s,0)]}{s(1+s)}. \tag{36}$$

A square-root dependence on time for small time intervals is described in Eq. (35) by the factor $(1+s^*)^{-2}\approx 1-2\sqrt{6t/\tau}$ for $\overline{\cos\theta}=0$. Note that this factor appears when one uses the image method and chooses the image reflection plane to coincide with the physical boundary.

The functions $g_0=g_0(t)$ and $J_M(s,t)$ are regular with respect to t. Their temporal dependence must be accounted for if one considers a wider temporal range. However, for higher values of $t\geq\tau$ Eq. (35) becomes invalid, since in this case the parameters g_j for $j>1$ are no longer negligible even for an isotropic phase function, corresponding to Eq. (17).

Note that Eqs. (34) and (35) for finite values of t correspond to the Milne solution for an absorbing system with nonunity albedo.

Now we turn to an analysis of Eq. (29) in the P_1 approximation. Substituting Eq. (29) into Eq. (26) and taking $s=0$, we get

$$c_1(t)=-\frac{3\beta(1-g_0)g_1}{s^*}[1+c_1(t)B_4(t)+c_2(t)B_3(t)], \tag{37}$$

where

$$B_n(t)=\frac{\beta}{2}\int_1^\infty\frac{ds_1}{s_1^n}\frac{\exp[-J(s_1,t)]}{(s_1+1)(s_1+s^*)}. \tag{38}$$

To verify the solution we have also used Eq. (26) at $s=s^*$ producing the same results. Since $B_n(t)$ are integrals of known functions, Eqs. (37) and (32) readily permit one to find the parameters $c_1(t)$ and $c_2(t)$ and thus calculate the temporal correlation function. For a larger time range, $2t/\tau \geqslant 1$, the binary permittivity correlation function $\langle\delta\varepsilon(0)\delta\varepsilon(t)\rangle$ must be known to calculate the parameters g_0 and g_1 without expansion of the exponential $\exp[-2(1-\overline{\cos\theta})t/\tau]$ in Eq. (16).

Consider the small time range $t/\tau\ll 1$. As is seen from Eq. (37), the parameter $c_1(t)$ turns out to be small of the order of $\sqrt{t}$. In fact, with $1-g_0\approx(2t/\tau)(1-\overline{\cos\theta})$ and the definition (30) for s^*, the factor before the square brackets in Eq. (37) can be written in first order in t as $3\beta(1-g_0)g_1/s^*\approx\overline{\cos\theta}\sqrt{18t/\tau}$. Then the term with $c_1(t)$ in the RHS of Eq. (37) can be omitted, and using the identity (A21) we calculate

$$c_1(t)\approx-\overline{\cos\theta}\sqrt{18t/\tau}\exp[-J_M(1,0)]. \tag{39}$$

Thus, taking into account Eq. (32), we obtain the temporal correlation function in the P_1 approximation as

$$C(t)\approx 1.5(1+s^*)^{-2}\exp[-2J(1,t)][1+(2/\sqrt{3})c_1(t)e^{J(1,t)}]$$
$$\approx 1.5\exp[-2J(1,t)][1-2s^*+2\,\overline{\cos\theta}\sqrt{6t/\tau}]. \tag{40}$$

Formula (40) is valid to first order in $\sqrt{t/\tau}$. Taking account of the numerical value [6] $1.5\exp[-2J_M(1,0)]\approx 4.227\,67$, it can be rewritten as follows:

$$C(t)\approx 4.227\,67\left(1-2\sqrt{\frac{6t}{\tau}}\right). \tag{41}$$

Thus the decay of the temporal correlation function appears to be universal in the t/τ variable for the small time range, in the P_1 approximation. The universality means that the initial slope in units of t/τ does not depend on the anisotropy magnitude $\overline{\cos\theta}$, with the slope coefficient being exactly 2. This appears to be in good agreement with the measurements of Ref. [19].

For better insight into what is causing this behavior of the correlation function we compare solution (40) with the corresponding formula from the diffusion approximation.

The usual approach to the boundary problem within the diffusion approximation consists in the image method. When one chooses the mirror image plane coincident with the physical boundary this method readily yields

$$C(t)\sim\frac{1}{(1+s^*)^2}\sim 1-2(1-\overline{\cos\theta})\sqrt{6t/\tau}.$$

This relationship describes the initial slope properly in the case of isotropic scattering, $\overline{\cos\theta}=0$, but becomes absolutely inadequate for the highly anisotropic case, $1-\overline{\cos\theta}\ll 1$, predicting a zero decay rate.

The diffusion approximation requires the image plane to be moved outside the medium at a distance $\frac{2}{3}l^*$, usually changed to $0.7104l^*$. This requirement is known to follow the energy conservation law. One obtains therewith [8]

$$C(t)\sim\frac{1}{(1+s^*)^2}\left\{1+\frac{1}{s^*}[1-\exp(-1.4208\sqrt{6t/\tau})]\right\}. \tag{42}$$

The single-scattering anisotropy influences the temporal correlation decay in two opposite ways. On the one hand, in both solutions, the Milne-like result Eq. (40) as well as Eq. (42), there appears a factor $(1+s^*)^{-2}$, which causes a slower decay with increasing $\overline{\cos\theta}$ due to the decrease of the parameter $s^*\approx\sqrt{6t/\tau}(1-\overline{\cos\theta})$.

This behavior of the parameter s^* is due, first, to an increase of the effective temporal parameter $\tau\rightarrow(D\overline{q^2})^{-1}=[2Dk^2(1-\overline{\cos\theta})]^{-1}$ with extension of the single-scattering indicatrix, and, secondly, to the diffusion mechanism of radiation transport at large distances, as described by Eq. (33).

On the other hand, the term in the square brackets in Eq. (42) and the term containing the parameter $c_1(t)$ in the Milne-like solution (40) both cause a more rapid decay of the temporal correlation function. The term in Eq. (42) appears because the mirror plane is moved outside the medium. One carries out this removal satisfying the energy flux conservation.

A similar decrease of the parameter $c_1(t)$ with increasing $\overline{\cos\theta}$, as is seen from Eq. (39), is caused by the fact that the RHS of the Milne equation Eq. (26) becomes zero at $s=0$ and $t=0$. In its turn this is due to the equation of balance of the incident and scattered radiation, as has been shown earlier [21].

Thus the appearance of terms increasing the slope of the temporal correlation with $\overline{\cos\theta}$ is caused in both cases by the conservation law. The crucial difference between the two solutions Eq. (40) and Eq. (42) is that in the Milne-like solution the terms exhibiting opposite dependencies on anisotropy cancel out exactly the dependence on $\overline{\cos\theta}$, while in the diffusion approximation the two similar terms fail to cancel out this dependence on $\overline{\cos\theta}$.

In Fig. 1 the temporal field correlation function calculated with Eqs. (29), (32), and (37) is plotted against time for different values of $\overline{\cos\theta}$. Note that in the case of strong anisotropy, $1-\overline{\cos\theta}\ll 1$, the parameters $1-g_0$ and s^* are still small even for values of t/τ of the order of unity; this makes Eq. (37) valid beyond the limit $t/\tau\approx 1$.

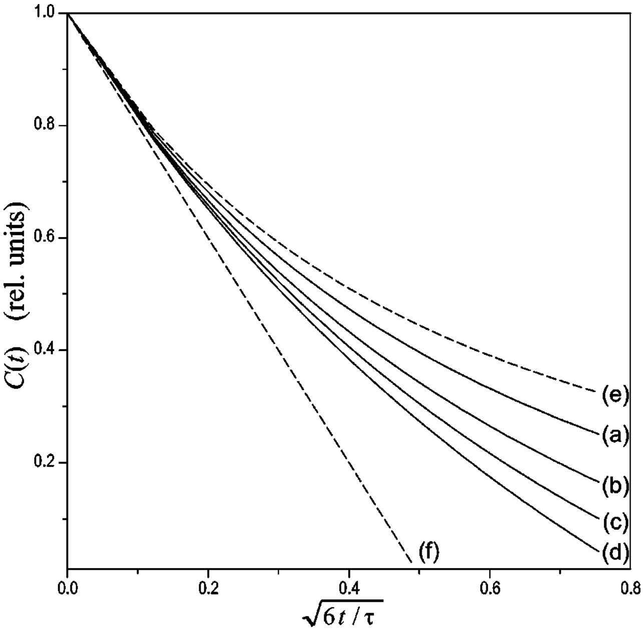

FIG. 1. The field temporal correlation function $C(t,1)$, normalized by the intensity in the P_1 approximation, vs $\sqrt{6t/\tau}$, for different $\overline{\cos\theta}$: (a) $\overline{\cos\theta}=0$, (b) $\overline{\cos\theta}=0.5$, (c) $\overline{\cos\theta}=0.75$, (d) $\overline{\cos\theta}=0.9$, (e) and (f) are fits $C(t,1)\sim(1+\sqrt{6t/\tau})^{-2}$ and $C(t,1)\sim(1-2\sqrt{6t/\tau})$, respectively, illustrating the initial slope universality.

The noted universality of the temporal correlation decay is seen to be violated with increasing time. As $\overline{\cos\theta}$ increases the decay rate of the correlation function increases also, within the P_1 approximation.

The principal qualitative merit of the diffusion approximation result (42) is that it predicts an initial temporal dependence of the form

$$C(t)\sim 1-\gamma\sqrt{6t/\tau},$$

where γ is finite for any value of $\overline{\cos\theta}$. However, this prediction appears to be invalid quantitatively. In particular, Eq. (42) gives $\gamma=2.4$ for $\overline{\cos\theta}=0$ and $\gamma=0.7$ for $\overline{\cos\theta}\to 1$. In the larger time range the diffusion approximation forecasts a decrease of the decay rate, strictly opposite to the exact result, which predicts a decay rate increase.

We calculate the temporal correlation function in the P_2 approximation also. Finding the parameters $c_0(t)$, $c_1(t)$, and $c_2(t)$, we substitute solution (29) into Eq. (26) and expand the latter into a series in s. Equating terms of the zeroth and first orders in s, respectively, and using Eq. (32), we get a closed algebraic system with respect to the sought parameters.

In Fig. 2 the results calculated in the P_2 approximation are shown for different $\overline{\cos^2\theta}$ values, keeping the parameter $\overline{\cos\theta}$ fixed at $\overline{\cos\theta}=0.6$. Including the parameter $P_2=\frac{1}{2}(3\,\overline{\cos^2\theta}-1)$ is seen to change the initial slope slightly and causes a slower damping of the temporal correlation function at larger times.

As is seen from Figs. 1 and 2, the parameters $\overline{\cos\theta}$ and $\overline{\cos^2\theta}$ change the decay rate of the temporal correlation function in opposite directions. With increasing $\overline{\cos\theta}$ the decay rate increases, and with increasing $\overline{\cos^2\theta}$ the decay rate decreases.

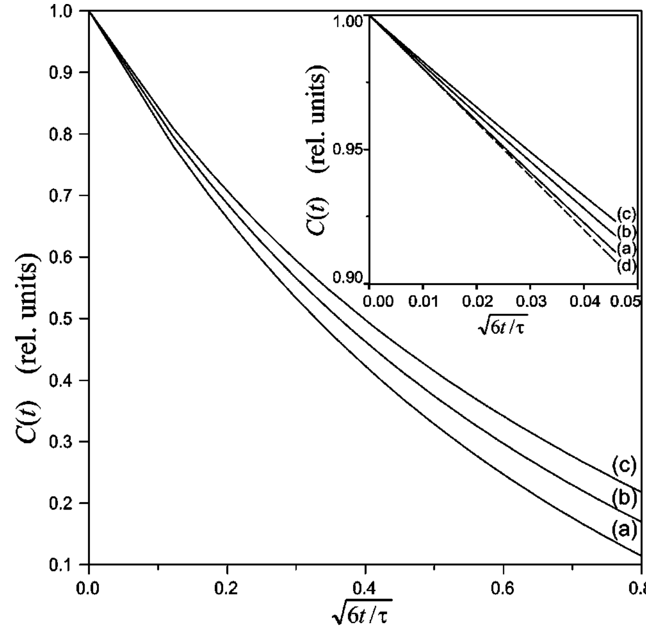

FIG. 2. The normalized field temporal correlation function in the P_2 approximation vs $\sqrt{6t/\tau}$ for $\overline{\cos\theta}=0.6$: (a) $P_2=0$, (b) $P_2=0.2$, (c) $P_2=0.36$. Dotted line (d) in the inset is a fit to Eq. (41).

To account for $\overline{\cos\theta}$ and $\overline{\cos^2\theta}$ simultaneously we use the Henyey-Greenstein phase function [10]

$$G\left(2k\sin\frac{\theta}{2}\right)\sim\frac{1-(\overline{\cos\theta})^2}{[1+(\overline{\cos\theta})^2-2\,\overline{\cos\theta}\cos\theta]^{3/2}},\qquad(43)$$

where $\frac{1}{2}(3\,\overline{\cos^2\theta}-1)=(\overline{\cos\theta})^2$. In Fig. 3 the results of a calculation are shown for several pairs of g_1 and g_2 values, which are chosen to correspond to the Henyey-Greenstein function. For better comparison with experiment we plotted the intensity correlation function $C_2(t)=\langle I(0)I(t)\rangle_c\approx|C(t)|^2$ against $\sqrt{t/\tau}$. We also reproduce the known measurement data from Ref. [19]. In the temporal range where dispersion of the data is small, the theoretical curves for different $\overline{\cos\theta}$ are seen to essentially coincide with the measured one. Note that in restricting oneself to the P_2 term in the phase function one bounds the P_2 values, $P_2<0.4$, which in turn limits the $\overline{\cos\theta}$ values to $\overline{\cos\theta}<0.63$.

Thus we conclude that by accounting simultaneously for two parameters of anisotropy within the Henyey-Greenstein model one obtains a temporal correlation decay law that turns out to be universal in the dimensionless variable t/τ in a wide temporal range. In turn, this universality means that specific peculiarities of a scattering system are contained solely in the parameter τ.

For $\sqrt{t/\tau}>0.35$ the curves with different values of $\overline{\cos\theta}$ are seen to diverge. However, the calculated results agree quantitatively with the experiment in this temporal range also, assuming a growing data scatter.

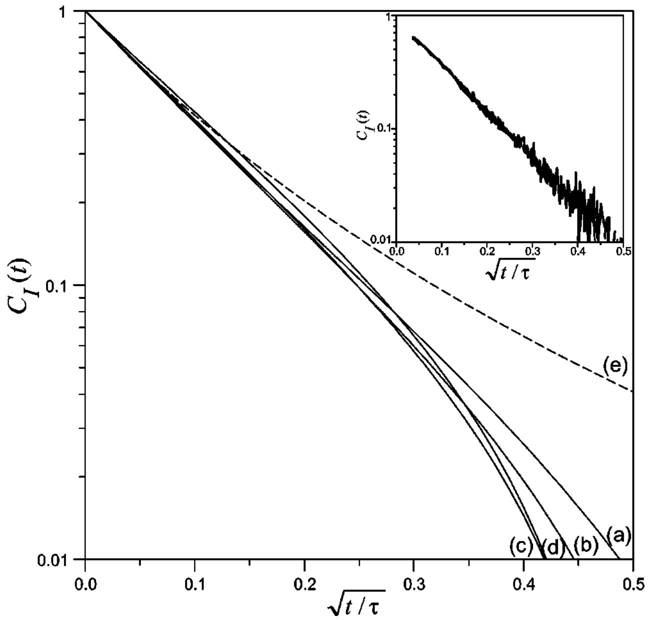

FIG. 3. The intensity correlation function vs $\sqrt{t/\tau}$ in the P_2 approximation with $g_2=(\overline{\cos\theta})^2$ in correspondence with Eq. (42): (a) $\overline{\cos\theta}=0$, (b) $\overline{\cos\theta}=0.2$, (c) $\overline{\cos\theta}=0.4$, (d) $\overline{\cos\theta}=0.6$; the inset represents the experimental plot of Ref. [19].

V. CONCLUSION

We have found the exact solution of the boundary problem for the field temporal correlation function accounting for single-scattering anisotropy up to the second-order Legendre polynomial. This solution covers an intermediate range of anisotropy between the pointlike scatterer system described by the Milne solution, and the solution found earlier [6,17] for the highly anisotropic case.

The main feature of the solution found is the universal initial slope of the temporal correlation function, independent of the anisotropy parameters in the case of Brownian diffusion. This result is exact in the P_1 approximation and turns out to be a consequence of the fact that the anisotropy effects appearing, on the one hand, due to the radiation diffusion and, on the other hand, due to the energy flux balance at the boundary cancel each other out exactly.

Since in the present consideration the expansion parameter is $t/\tau(1-\overline{\cos\theta})$, which remains small in the case of high anisotropy even for t/τ of the order of unity, the theory might be applied in an expanded temporal range.

The present results permit one to obtain the specific parameter $\tau=(Dk^2)^{-1}$ from measurement data since the formula achieved does not contain any fitting parameters. This makes it desirable to have systematic measurements of the temporal correlation function in suspensions in wide ranges of concentration and scatterer size.

Note that in the case of different mechanisms of permittivity fluctuation decay other than Brownian diffusion, for instance, in the case of relaxation, when the fluctuation decay is independent of the wave vector transfer, we should obtain a different dependence on $\overline{\cos\theta}$.

We hope that the method developed here can be applied to other boundary problems of multiple scattering, such as coherent backscattering.

APPENDIX

To eliminate the pole $s=0$ in the function $\tilde{\gamma}(s,t)$ and poles $s=0$ and $s=-1$ in the RHS of Eq. (26) we multiply this equation by $s^2(s+1)$. The function $\psi(s,t)$ is seen to be regular in the plane of the complex variable s with two cuts along the real axis, $s\geqslant 1$ and $s\leqslant -1$, respectively. We define the zero of $\psi(s,t)$ as the root $s=s^*$ of the transcendental equation

$$s^{*2}=\frac{(1-g_0)\left\{1-3g_1m_1(s^*)-\frac{15}{4}(1-g_1)g_2[3m_2(s^*)-m_1(s^*)]\right\}}{g_0m_1(s^*)-\frac{5}{4}g_2[3m_2(s^*)-m_1(s^*)]}. \tag{A1}$$

The parameter β is defined as follows:

$$\beta^{-2}=\lim_{s\to 0}\frac{\psi(s,t)}{s^{*2}-s^2}=\frac{(1-g_0)(1-g_1)(1-g_2)}{s^{*2}}. \tag{A2}$$

To first order in $1-g_0\ll 1$ Eqs. (A1) and (A2) give Eq. (30).

We define the function

$$\psi_N(s,t)=\psi(s,t)\frac{\beta^2}{s^{*2}-s^2}\left(1-\frac{s^2}{\beta^2}\right) \tag{A3}$$

with the asymptotic properties $\psi_N(0,t)=1$ and $\psi_N(s,t)\to 1$ for $s\to\infty$. Equation (26) can be rewritten as follows:

$$\psi_N(s,t)\gamma^+(s,t)=\gamma^-(s,t), \tag{A4}$$

where

$$\gamma^+(s,t)=\frac{(s^*+s)s^2(s+1)\tilde{\gamma}(s,t)}{\beta+s}, \tag{A5}$$

$$\gamma^{-}(s,t)=\frac{s^{2}(s+1)(\beta-s)A(s,t)}{s^{*}-s}. \tag{A6}$$

The function $\tilde{\gamma}^{+}(s,t)$ is regular in the right half plane $\mathrm{Re}\,s \geqslant 0$, as is seen from the definitions (20) and (23), and $\tilde{\gamma}^{-}(s,t)$ is regular in the left half plane $\mathrm{Re}\,s<1$, according to Eq. (22). Thus $\psi_N(s,t)$ turns out to be a ratio of functions regular at $\mathrm{Re}\,s\geqslant 0$ and at $\mathrm{Re}\,s<1$, respectively,

$$\psi_N(s,t)=\frac{\gamma^{-}(s)}{\gamma^{+}(s)}. \tag{A7}$$

Since $\psi_N(s,t)$ is an odd function of s, $\psi_N(s,t)=\psi_N(-s,t)$, it can be presented in the form

$$\psi_N(s,t)=\frac{h^{+}(s)}{h^{-}(s)}, \tag{A8}$$

where $h^{+}(s)$ is regular at $\mathrm{Re}\,s>-1$ and $h^{-}(s)$ is regular at $\mathrm{Re}\,s\leqslant 0$. One can put $h^{+}(0)=h^{-}(0)=1$ since asymptotically $\psi_N(0,t)=1$. We assume also that $h^{+}(s)$ does not contain zeros in the right half plane $\mathrm{Re}\,s>0$, and nor does $h^{-}(s)$ in the left one $\mathrm{Re}\,s<0$. Otherwise these functions should be redefined, considering the finite number of left-hand side zeros s_j^{-} of the function $h^{-}(s)$ as poles of the function $h^{+}(s)$ in the left half plane, and, contrarily, the zeros s_j^{+} of the function $h^{+}(s)$ as poles of the function $h^{-}(s)$ in the right half plane. Consider the integral

$$I(s)=-\frac{s}{2\pi i}\int_{-i\infty}^{i\infty}\frac{ds'}{s'(s'-s)}\ln \psi_N(s',t). \tag{A9}$$

Substituting Eq. (A8) into Eq. (A9) and calculating the integral with the residue theorem, we obtain

$$I(s)=\begin{cases}\ln h^{+}(s), & \mathrm{Re}\,s>0,\\ \ln h^{-}(s), & \mathrm{Re}\,s<0.\end{cases} \tag{A10}$$

Taking into account the symmetry of $\psi_N(s)$ with respect to the sign of s, we get

$$h^{-}(-s)=\frac{1}{h^{+}(s)}, \quad \mathrm{Re}\,s>0. \tag{A11}$$

From Eqs. (A7) and (A8) it follows that

$$h^{-}(s)\gamma^{-}(s,t)=h^{+}(s)\gamma^{+}(s,t). \tag{A12}$$

In the limit $s\to\infty$ we have asymptotically that $h^{\pm}(s)\sim$const, and $\gamma^{+}(s)\sim s^{2}$, $\gamma^{-}(s)\sim s^{2}$. Thus we can easily obtain

$$h^{\pm}(s)\gamma^{\pm}(s)\sim s^{2} \quad \text{at} \quad s\to\infty. \tag{A13}$$

Since the RHS of Eq. (A12) is regular at $\mathrm{Re}\,s\geqslant 0$, and the LHS at $\mathrm{Re}\,s\leqslant 0$, they are both regular in the whole plane. According to the generalized Liouville theorem a function regular everywhere can be presented, taking into account the asymptote (A13), as a second-order polynomial in s:

$$h^{-}(s)\gamma^{-}(s)=h^{+}(s)\gamma^{+}(s)=c_0+c_1 s+c_2 s^{2}, \quad c_j=c_j(t). \tag{A14}$$

It follows from definitions (22) that the function $\gamma^{-}(s)$ is known at $s=-1$:

$$\gamma^{-}(-1)=\frac{1+\beta}{1+s^{*}}\left\{g_0+3g_1(1-g_0)-\frac{5}{2}g_2[1-3(1-g_0)\times(1-g_1)]\right\}. \tag{A15}$$

Putting $s=-1$ in Eq. (A14) and using Eq. (A11), we get

$$c_0-c_1+c_2=\gamma^{-}(-1)h^{-}(-1)=\frac{\gamma^{-}(-1)}{h^{+}(+1)}. \tag{A16}$$

Equation (A14) yields

$$\gamma^{+}(s,t)=(c_0+c_1 s+c_2 s^{2})\frac{1}{h^{+}(s)} =(c_0+c_1 s+c_2 s^{2})\exp[-I(s,t)]. \tag{A17}$$

Taking into account the relationship (A11) and explicitly taking the integral

$$-\frac{s}{2\pi i}\int_{-i\infty}^{i\infty}\frac{ds'}{s'(s'-s)}\ln\left(1-\frac{s'^{2}}{\beta^{2}}\right)=\ln\left(1+\frac{s}{\beta}\right), \quad \mathrm{Re}\,s>0, \tag{A18}$$

we get

$$\exp[-I(s,t)]=\frac{1}{1+s/\beta}\exp[-J(s,t)], \quad \mathrm{Re}\,s>0. \tag{A19}$$

Substituting Eqs. (A5) and (A19) into Eq. (A17), we come to Eq. (29). Equations (A15), (A16), and (A19) yield Eq. (32).

Substituting the solution (29) into Eq. (26) and expanding the latter in a series in s, we obtain a sequence of identities to be satisfied obligatorily, which relate the integral parameters $B_n(t)$. In particular, for isotropic single scattering, $g_1=g_2=0$, at $t=0$ i.e., for the Milne solution case, one obtains

$$c_2^{M}B_2^{M}(0)=1,$$

$$B_2^{M}(0)+B_3^{M}(0)=\frac{1}{\sqrt{3}}, \tag{A20}$$

$$B_3^{M}(0)+B_4^{M}(0)=\frac{z^{*}}{\sqrt{3}},$$

where the parameter z^{*} is a linear term in the expansion $\exp[-J_M(s,0)]\approx 1+sz^{*}+O(s^{2})$,

$$z^* = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{ds_1}{s_1^2}\ln[3m_1(is_1)] = 0.7104,$$

and is the known extrapolation parameter of the Milne theory.

The numerical values of the integrals (38) are found to be $B_2^M(0) = 0.3427$, $B_3^M(0) = 0.2343$, and $B_4^M(0) = 0.1787$. They are seen to satisfy Eqs. (A20) up to the third sign. Combining the first two equations (A20) and noticing that

$$c_2^M(0) = \sqrt{3}\,\exp[-J_M(1,0)]$$

from Eq. (32), we easily find that

$$1 + c_2^M B_3^M(0) = \exp[-J_M(1,0)]. \qquad \text{(A21)}$$

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